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THE LOG-A-POSTERIORI PROBABILITY METRIC FOR USE IN SEQUENTIAL DECODING

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ABSTRACT

Convolutional encoding with sequential decoding is one of the best coded communications systems available for spacecraft telemetry. When implementing a sequential decoder, a decoding confidence measure based upon the received signal probabilities is used. The best known measure is the "log-a-posteriori probability metric." This paper describes and derives this metric in detail. Tabulations of the metric for various telemetry system parameters are also given. Both a continuous and a quantized metric are treated for various symbol signal-to-noise ratios on a Gaussian channel. Quantized symbol metrics are tabulated for 2, 4, 8, and 16 levels, and it is shown that the quantized LAP metric is an approximation of the ideal continuous metric.

CONTENTS

Abstract	ii
INTRODUCTION	1
THE CHANNEL MODEL	2
CHANNEL CONDITIONAL PROBABILITIES	3
THE SEQUENTIAL DECODER LAP METRIC	3
THE CONTINUOUS SYMBOL LAP METRIC	6
THE QUANTIZED SYMBOL METRIC	7
COMPARISON OF CONTINUOUS AND QUANTIZED METRICS	10
TYPICAL METRIC BEHAVIOR IN A SEQUENTIAL DECODER	10
CONCLUSIONS	12
References	13
Appendix A—Statistics at Output of a PCM Matched Filter	15
Appendix B—Tabulation of the Symbol Mutual Information Function	21

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INTRODUCTION

During sequential decoding on a continuous, binary-symbol, communications channel, each received symbol has two weights. One weight is a measure of confidence that a binary "1" was transmitted, and the other is a measure of confidence that a binary "0" was transmitted. If the weights are proportional to the symbol's matched-filter correlation voltage, then the decoder is said to use a "correlation metric." A better measure, however, is one in which the weights are proportional to the probability that a 1 or 0 was sent, given the correlation voltage (References 1 and 2). This is called an "a-posteriori-probability metric," and variations of it are among the best known for use in sequential decoding.

Since the sequential decoding technique itself is well described in the literature, no attempt is made to do so here. It is sufficient to say that a sequential decoder, whether it be implemented with a general-purpose computer or special-purpose hardware, must store a metric table for use by the decoding algorithm, and that the decoder's performance is sensitive to the choice of this metric.

When an engineer attempts to implement a sequential decoder, he finds that the log-a-posteriori-probability (LAP) metric is only briefly described in the literature. No derivations are given, and only the metric for a particular set of conditions is given. This paper provides a complete step-by-step derivation of the LAP metric, and tabulations of the metric are given for a variety of symbol signal-to-noise ratios and quantization levels. Furthermore, it is shown graphically that the quantized LAP metric is an approximation to the continuous metric.

In the following paragraphs, the channel characteristics for a space data modem are defined, and the LAP metric is derived for a memory-less channel with arbitrary noise probability densities and for the continuous LAP metric assuming a gaussian noise channel. Because an A-to-D conversion is ordinarily performed at the receiver, a quantized LAP metric is described and tabulated. This quantized metric is then compared with the continuous metric. Finally, system operation and the behavior of the decoder's branch metric are described with a specific example.

THE CHANNEL MODEL

The mathematical model of the communications channel assumed is shown in Figure 1. For each data symbol d_i entering the encoder, " v " binary symbols are transmitted. The transmitter sends plus or minus \sqrt{S} volts for a 1 or 0 symbol, respectively. The modulation power

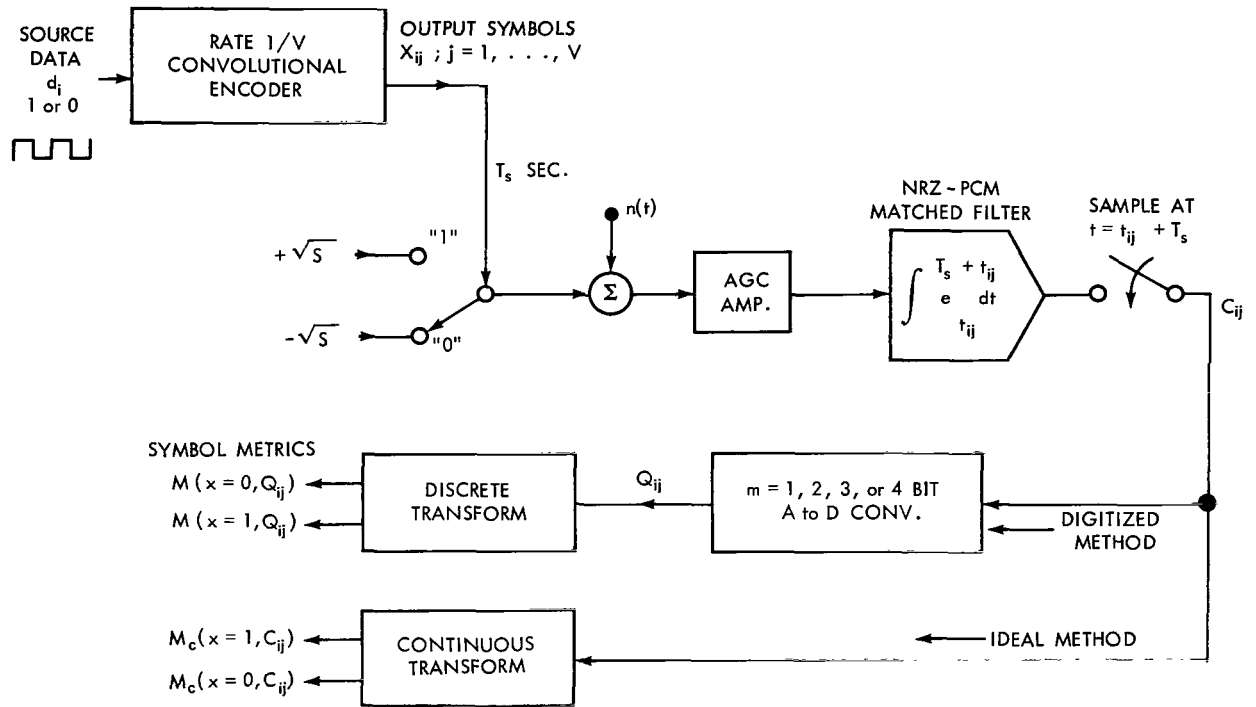


Figure 1—Communications system model.

is then S watts. Additive noise $n(t)$ has N_0 watts per Hz single-sided power spectral density. A gain-controlled amplifier in the receiver maintains a constant signal amplitude into the filter. The matched filter is sampled at the end of a symbol period, giving a correlation voltage C_{ij} .

Indices " i " and " j " denote the j^{th} symbol associated with the i^{th} data bit. The correlation C_{ij} is a continuous, random variable because of the additive gaussian noise.

Given each C_{ij} , the decoder hypothesizes the transmission of a 1 or 0. Let \hat{x} denote this hypothesis; the symbol metrics $M_c(\hat{x} | C_{ij})$ are then summed by the decoder to compute a branch metric. It is impractical to use continuous values in a digital machine; therefore the C_{ij} correlations are usually digitized. Using more than about 16 levels (4 bit A-to-D) gains little in system performance (Reference 2). The quantized correlations Q_{ij} are then used to find the quantized metric $M(\hat{x} | Q_{ij})$.

CHANNEL CONDITIONAL PROBABILITIES

The symbol's correlation voltage C_{ij} will have a gaussian probability density since the receiver's amplifier and filter are linear devices. The mean-to-standard deviation ratio of C_{ij} is shown in Appendix A to be

$$\frac{\mu}{\sigma} = \sqrt{2 \frac{ST_s}{N_0}}, \quad (1)$$

where

μ = mean value of C_{ij} ,

σ = standard deviation of C_{ij} ,

S = modulation power in watts,

T_s = duration of binary symbol in sec,

N_0 = noise density of $n(t)$ in watts per Hz.

If x_{ij} is a binary 1, then μ is some positive voltage; when x_{ij} is a 0, then μ is the same voltage of negative value. Let the gain-controlled amplifier adjust this voltage magnitude to 1 volt. Then the conditional densities of C_{ij} are

$$p(C_{ij} | x_{ij} = 1) = \text{Normal} \left[\mu = +1, \sigma = \left(2 \frac{ST_s}{N_0} \right)^{-1/2} \right] \quad (2)$$

and

$$p(C_{ij} | x_{ij} = 0) = \text{Normal} \left[\mu = -1, \sigma = \left(2 \frac{ST_s}{N_0} \right)^{-1/2} \right] \cdot (3)$$

Figure 2 shows these conditional densities.

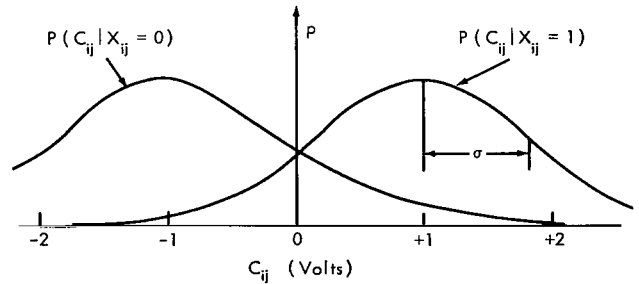


Figure 2—Conditional probability densities for the symbol matched-filter samples.

THE SEQUENTIAL DECODER LAP METRIC

A sequential decoder is basically a conditional probability computer. In its algorithmic search for the correct path or branch in the coding tree, it computes the probability that the hypothesized

symbols \hat{x}_{ij} could have been transmitted, given the received c_{ij} . This conditional branch probability is

$$P(\underbrace{\hat{x}_{1,1}, \hat{x}_{1,2}, \dots, \hat{x}_{k,v}}_{\hat{X}_n} | \underbrace{c_{1,1}, c_{1,2}, \dots, c_{k,v}}_{C_n}) , \quad (4)$$

where

\hat{x}_{ij} = the binary symbols hypothesized by the decoder for this branch,

k = the number of information bits shifted into the encoder to generate the branch,

v = the number of encoder symbols transmitted per input information bit, and

c_{ij} = the received symbol correlation voltages.

To simplify notation, replace the \hat{x}_{ij} and c_{ij} sequences in Equation 4 with \hat{X}_n and C_n , respectively. The "n" subscript represents a one-dimensional serial ordering of the sequences and furthermore implies $n = kv$. Using the same notation, X_n represents the actual transmitted sequence of kv binary symbols. The conditional branch probability, using hypothesized symbols, is

$$\triangleq P(\hat{X}_n | C_n) . \quad (5)$$

Equation 5 is the hypothesized "a-posteriori branch probability," which is a measure of confidence that $\hat{X}_n = X_n$. For $v \geq 2$, a large k , and not-too-high noise, we expect $P(\hat{X}_n | C_n)$ to be near unity only if $\hat{X}_n = X_n$. Otherwise, it should be small for "good" convolutional codes. An optimum decoder selects \hat{X}_n so as to maximize this probability.

The a-posteriori branch probability for true transmitted sequence X_n may be found using Bayes Theorem:

$$P(X_n | C_n) = \frac{P(X_n, C_n)}{P(C_n)} = \frac{P(C_n | X_n) P(X_n)}{P(C_n)} . \quad (6)$$

Since the channel model has no memory from symbol to symbol, the symbol probabilities are independent. Thus, the sequence conditional probability in Equation 6 may be expressed as the product of the symbol probabilities:

$$\frac{P(C_n | X_n) P(X_n)}{P(C_n)} = P(X_n) \prod_{m=1}^{m=n=kv} \frac{p(c_m | x_m)}{p(c_m)} . \quad (7)$$

The $P(X_n)$ is just the branch probability. Each branch is a-priori equiprobable with random data, and there are 2^k unique branches in an $n(= kv)$ symbol code tree. Therefore,

$$P(X_n) = 2^{-k} . \quad (8)$$

To simplify the decoder's implementation, an additive measure of probability is desirable. The logarithm (base = 2) of $P(X_n | C_n)$ allows this without disturbing the monotonic nature of the conditional probability. Note that very small branch probabilities become large negative numbers.

Combining Equations 6, 7, and 8; taking the logarithm; and using the double subscript symbol indexing gives

$$\log_2 P(X_n | C_n) = -k + \sum_{i=1}^k \sum_{j=1}^v \log_2 \frac{p(C_{ij} | x_{ij})}{p(C_{ij})} .$$

Of course, the actual x_{ij} sequence is not known, and the decoder must hypothesize these symbols \hat{x}_{ij} so as to maximize this probability. Thus, substituting \hat{x}_{ij} and letting $k = kv/v$ so that it may be combined in the summation gives

$$\log_2 P(\hat{X}_n | C_n) = \underbrace{\sum_{i=1}^k \left[\underbrace{\sum_{j=1}^v \left\{ \log_2 \left[\frac{p(C_{ij} | \hat{x}_{ij})}{p(C_{ij})} \right] - \frac{1}{v} \right\}}_{\text{Symbol LAP Metric}} \right]}_{\text{Node LAP Metric}} \underbrace{\quad}_{\text{Branch LAP Metric}} . \quad (9)$$

Thus, the decoder computes the symbol-log-a-posteriori probability metrics and sums them over a node ($= v$ symbols). Information bit hypotheses are based on this node metric. The sum of the node metrics then forms the branch LAP metric.

The quantity

$$\log_2 \frac{p(C | \hat{x})}{p(C)}$$

in Equation 9 is called the mutual information between x and C , and the quantity $1/v$ is the convolutional code rate. The channel conditional probabilities, $p(C | \hat{x} = 1)$ and $p(C | \hat{x} = 0)$, were given in the paragraph entitled Channel Conditional Probabilities.

Define

$$\log_2 \frac{p(C | \hat{x})}{p(C)} \triangleq I(\hat{x}, C) \quad (10)$$

and the continuous symbol LAP metric $\triangleq M_c(\hat{x} | C_{ij})$; then it follows that

$$M_c(\hat{x} | C_{ij}) = I(\hat{x}, C_{ij}) - \frac{1}{v}. \quad (11)$$

It is clear from Equation 11 that it is only necessary to evaluate $I(\hat{x}, C)$ for the channel. The symbol metric M_c may be found then for any code rate by simply subtracting $1/v$.

THE CONTINUOUS SYMBOL LAP METRIC

The symbol LAP metric $M_c(x | C_{ij})$ is a continuous function of C . As noted earlier, it is easier to implement when digitized to as few bits as possible. However, it is useful to compare the continuous metric to the quantized metric to see how "close" the latter approximates it.

Define x_0 to mean $\hat{x} = 0$ and x_1 to mean $\hat{x} = 1$; then using Equation 10,

$$I(x_1, C) = \log_2 \frac{p(C | x_1)}{p(C)} = \log_2 \frac{p(C | x_1)}{p(C | x_0)p(x_0) + p(C | x_1)p(x_1)}.$$

For random data and linear parity check codes, it is generally true that $p(x_0) = p(x_1) = 1/2$. Therefore,

$$I(x_1, C) = 1 - \log_2 \left[\frac{p(C | x_0)}{p(C | x_1)} + 1 \right]. \quad (12)$$

Likewise,

$$I(x_0, C) = 1 - \log_2 \left[\frac{p(C | x_1)}{p(C | x_0)} + 1 \right]. \quad (13)$$

Using the conditional densities, Equations 2 and 3,

$$\begin{aligned} I(x_1, C) &= 1 - \log_2 \left\{ \frac{\exp \left[-\frac{(C + \mu)^2}{2\sigma^2} \right]}{\exp \left[-\frac{(C - \mu)^2}{2\sigma^2} \right]} + 1 \right\} \\ &= 1 - \log_2 \left[\exp \left(-\frac{2\mu C}{\sigma^2} \right) + 1 \right]. \end{aligned} \quad (14)$$

Similarly,

$$I(x_0, C) = 1 - \log_2 \left[\exp \left(+ \frac{2\mu C}{\sigma^2} \right) + 1 \right] \quad (15)$$

Since $I(x_0, C) = I(x_1, -C)$, it is only necessary to calculate $I(x_1, C)$. With $\mu = 1$ volt, then $\sigma = (2ST_s/N_0)^{-1/2}$ (from Equation 1), and finally,

$$I(x_1, C) = 1 - \log_2 \left[\exp \left(-4C \frac{ST_s}{N_0} \right) + 1 \right] \quad (16)$$

Figure 3 is a plot of Equation 16 for ST_s/N_0 values useful for $v = 2$ and 4 codes. Appendix B is a tabulation of Equation 16 for ST_s/N_0 values of -6 db through +3 db.

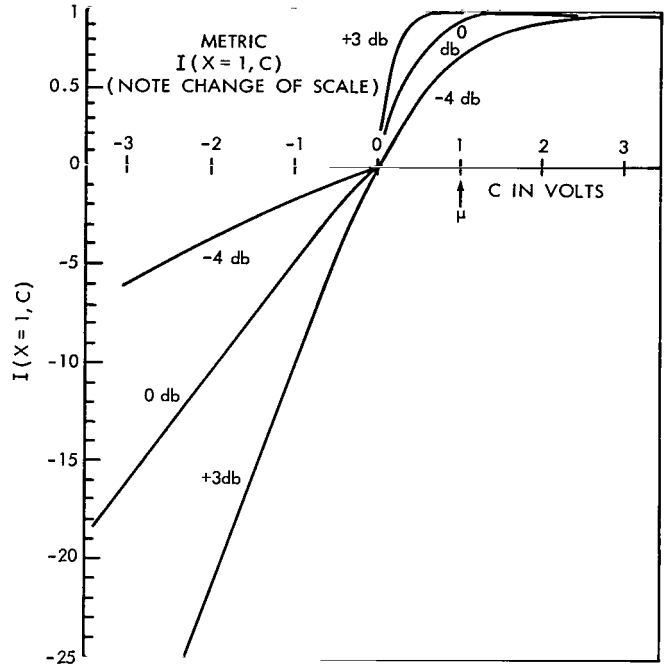


Figure 3—The continuous symbol metric for several ST_s/N_0 ratios.

THE QUANTIZED SYMBOL METRIC

As shown in Figure 1, an A-to-D conversion of a symbol correlation C gives rise to a number Q representing one of a set of 2^M quantiles of C . The quantized symbol LAP metric is defined on Q as

$$M(\hat{x}_{ij} | Q_{ij}) = \log_2 \left[\frac{P(Q_{ij} | \hat{x}_{ij})}{P(Q_{ij})} \right] - \frac{1}{v} \quad (17)$$

This is similar to Equation 9, but is a ratio of quantile probabilities rather than densities. A result analogous to Equation 12 can be readily obtained:

$$I(x_1, Q) = 1 - \log_2 \left[\frac{P(Q | x_0)}{P(Q | x_1)} + 1 \right] \quad (18)$$

where,

$$P(Q | x_i) = \int_Q \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ - \frac{[z + \mu(x_i)]^2}{2\sigma^2} \right\} dz$$

and

$$\mu(x_i) = \begin{cases} +1 & \text{if } x_i = x_1 \\ -1 & \text{if } x_i = x_0 \end{cases};$$

finally,

$$M(\hat{x} | Q) = I(x, C) - \frac{1}{v}. \quad (19)$$

Before calculation can proceed, there are several practical questions that must be answered:

1. How many A-to-D bits (m) should be used?
2. Over what range of C should the quantization levels be distributed?

Jacobs has shown (Reference 2) that the signal-to-noise ratio (SNR) loss between the use of 16 levels and an infinite number is under 0.1 db. The use of only two levels, however, incurs a 2-db loss. Selecting the number of levels is a system design problem. The metric is found in this paper for 2, 4, 8, and 16 levels.

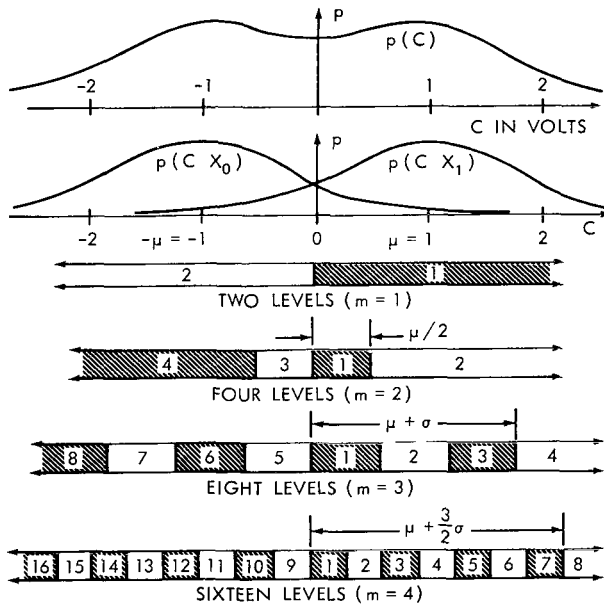


Figure 4—Quantile assignment nomogram.

Assigning the optimum level is difficult, since the exact cost function for the decoder's performance is not known. The assignments selected for this paper were chosen to give a good approximation to the continuous metric over all C where $p(C)$ was non-negligible. It is known from previous simulations that the level assignments are not critical.* Figure 4 summarizes the assumptions made. Note that the 8- and 16-level assignments are a function of σ and hence of the particular symbol SNR selected. The ordinary A-to-D converter constraint of equally spaced quantization intervals is assumed.

Table 1 lists the quantized mutual information, $I(x_1, Q)$, for four symbol SNR's. Using Equation 19, the LAP metric may be easily found, given the code rate.

*Private communication with G. David Forney, Jr., October 1967.

Table 1

Quantized Mutual Information, $I(x = 1, Q)$, Tabulation
(See Figure 3 for quantile assignments).

Quantile Number, Q	$ST_s/N_0 = -4 \text{ db}$ $\mu = 1, \sigma = 1.121$				$ST_s/N_0 = -1 \text{ db}$ $\mu = 1, \sigma = 0.793$			
	Number of Levels				Number of Levels			
	2	4	8	16	2	4	8	16
1	0.703	0.255	0.340	0.201	0.843	0.450	0.510	0.311
2	-1.42	0.818	0.743	0.510	-2.38	0.944	0.908	0.702
3		-0.31	0.908	0.712		-0.66	0.985	0.881
4		-2.07	0.976	0.835		-3.70	0.998	0.954
5			-0.45	0.908			-0.80	0.983
6			-1.61	0.948			-3.02	0.993
7			-3.02	0.972			-5.56	0.998
8			-4.91	0.989			-8.53	0.999
9				-0.23				-0.40
10				-0.80				-1.42
11				-1.47				-2.66
12				-2.21				-4.00
13				-3.01				-5.39
14				-3.84				-6.79
15				-4.69				-8.21
16				-6.04				-10.17
	$ST_s/N_0 = 0 \text{ db}$ $\mu = 1, \sigma = 0.707$				$ST_s/N_0 = +3 \text{ db}$ $\mu = 1, \sigma = 0.501$			
	2	4	8	16	2	4	8	16
1	0.992	0.532	0.578	0.358	0.966	0.788	0.788	0.539
2	-2.67	0.968	0.945	0.767	-4.45	0.997	0.994	0.925
3		-0.85	0.994	0.923		-1.87	0.999	0.989
4		-4.52	0.999	0.975		-8.27	1.000	0.998
5			-0.98	0.992			-1.87	0.999
6			-3.73	0.996			-7.01	1.000
7			-6.80	0.999			-12.38	1.000
8			-10.26	1.00			-18.04	1.000
9				-0.47				-0.87
10				-1.75				-3.31
11				-3.26				-6.06
12				-4.88				-8.88
13				-6.54				-11.70
14				-8.21				-14.53
15				-9.89				-17.35
16				-12.12				-20.71

The symbol SNR's, ST_s/N_0 , in Table 1 have been selected especially for rate 1/2 and 1/4 codes. With the E_{BIT}/N_0 * decoding thresholds defined as 3 db and 2 db for rate 1/2 and 1/4 codes respectively, the corresponding symbol SNR's are as follows:

code rate		Threshold	Threshold +3 db
1/2	1/2	0 db	3 db
	1/4	-4 db	-1 db

Since most operational telemeters normally operate above the system's threshold, SNR's 3 db above threshold are also given.

COMPARISON OF CONTINUOUS AND QUANTIZED METRICS

If the quantized metric is superimposed on the continuous metric, as shown in Figure 5, it is apparent that the quantized metric is a good stepwise approximation to the continuous metric.[†] Should a quantized metric be desired with different SNR's, etc., one could readily use a graphical approximation to the continuous metric.

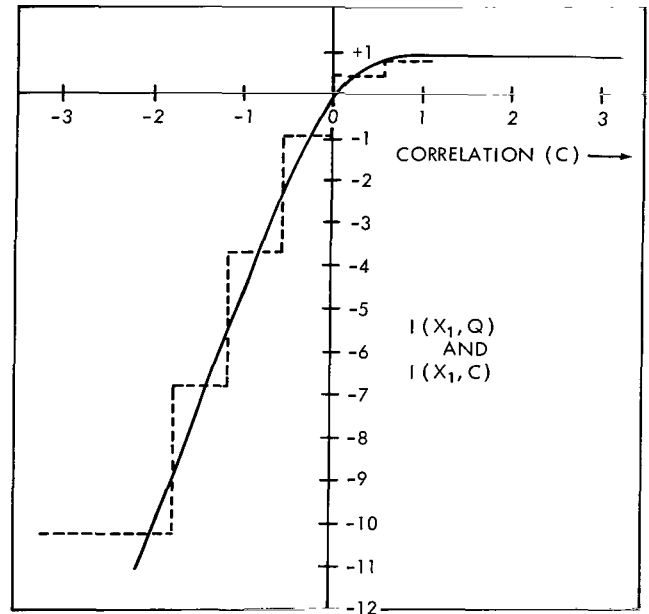


Figure 5—Comparison of continuous and 8-level quantized mutual information ($ST_s/N_0 = 0$ db).

TYPICAL METRIC BEHAVIOR IN A SEQUENTIAL DECODER

For a better appreciation of the behavior of a sequential decoder's branch metric when decoding a noisy signal, an example is given, based upon a computer simulation of a decoding system.[‡] This example consists of a time record of system parameters during the propagation of seven information bits through the encoder, channel, filter, and decoder.

An 8-level, quantized LAP metric is assumed; its assumptions and values are listed in Table 2. For efficient computer implementation, integer metric table values are desired. The $M(X|Q)$ values have been multiplied by 20, and the nearest integer value taken as the table value.

A time record of a rate 1/2 convolutional coded-sequential decoded system using this metric is shown in Figure 6. Each data bit d_i that enters the system causes two symbols to be transmitted,

* E_{BIT}/N_0 = Received energy per information bit/noise power spectral density.

†Metric and mutual information are used interchangeably since their only difference is the addend $-1/v$.

‡Saliga, T. V., "A Comparison of Sequential Decoding Metrics by Computer Simulation," to be published as a NASA Technical Report, 1968.

Table 2

Metric Table for Example (8-Level LAP Metric for $\mu/\sigma = \sqrt{2}$ or $ST_s/N_0 = 0$ db).

$M(\hat{x} = 0 Q)$	Decoder Table Value	Q	$M(\hat{x} = 1 Q)$	Decoder Table Value
-1.48	-30	1	0.078	2
-4.23	-85	2	0.445	9
-7.3	-146	3	0.494	10
-10.76	-216	4	0.500	10
0.078	2	5	-1.48	-30
0.445	9	6	-4.23	-85
0.494	10	7	-7.3	-146
0.500	10	8	-10.76	-216

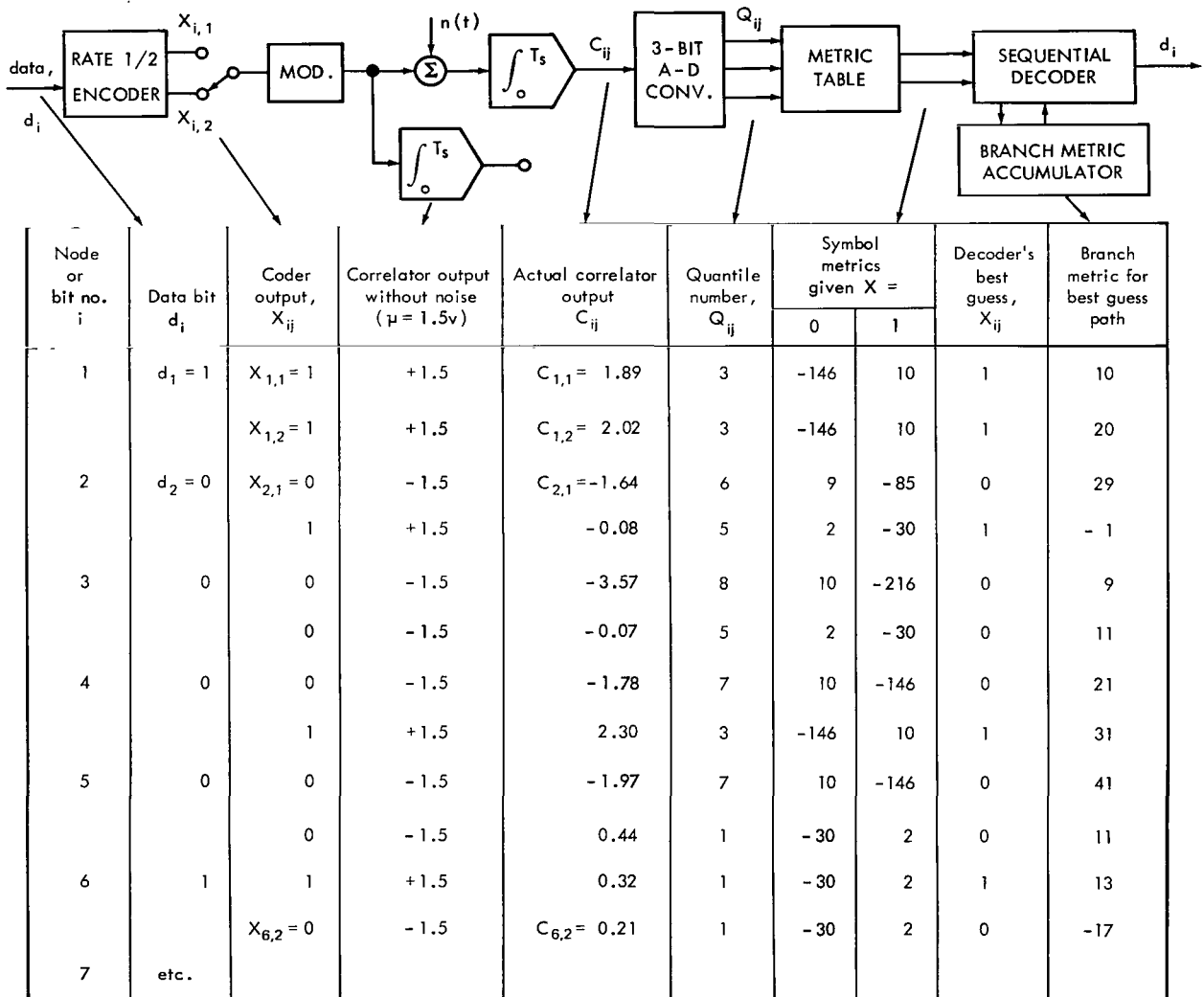


Figure 6—Operation analysis example.

$x_{i,1}$ and $x_{i,2}$. This in turn yields a pair of symbol correlation voltages, $C_{i,1}$ and $C_{i,2}$. Each of these A-to-D converted values, $Q_{i,1}$ and $Q_{i,2}$, interrogates the metric table for a pair of metric values. Depending on the decoder's algorithm and encoder's connections, two metric values are summed to form a node metric for a $\hat{d}_i = 1$ hypothesis. Similarly, the alternate metric pair are summed to form the node metric for the $\hat{d}_i = 0$ hypothesis. The "best guess" node metric is added into an accumulator to form the current branch metric, and its associated data bit \hat{d}_i is stored in a local data register.

A plot of the behavior of the branch LAP metric versus bit number is shown in Figure 7. If no bit errors are made and the noise is not too large, then the branch metric will increase slowly with "i". With one or more bit errors, it will tend to decrease rapidly in value as "i" increases.

CONCLUSIONS

A sequential decoder's performance is quite sensitive to the choice of a metric.* Should a proposed system have channel amplitude statistics significantly different from gaussian, then a new metric table should be found using general results (Equations 11, 12, 13, 17 and 18). This should insure that the decoder's error probability and overflow probabilities are low.

The author is not aware of any proof that the LAP metric is the best metric to use in a sequential decoder. However, it is the best of the known metrics.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, August 16, 1968
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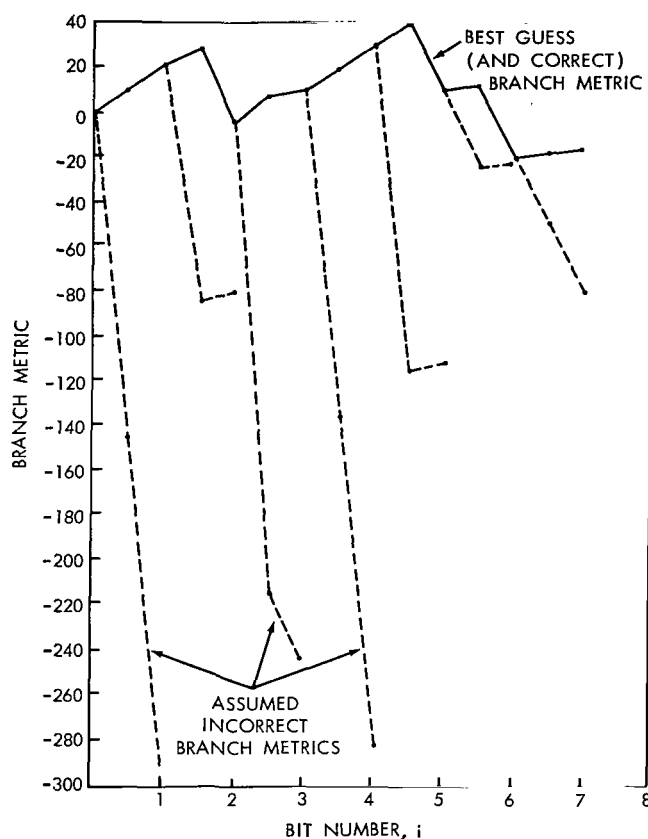


Figure 7—Branch metric behavior with a noisy signal.

*Saliga, T. V., "A Comparison of Sequential Decoding Metrics by Computer Simulation," to be published as a NASA Technical Report, 1968.

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Appendix A

Statistics at Output of a PCM Matched Filter

It is well known that a matched filter and a cross-correlator have identical output signal-to-noise ratios when the correlator's local signal is a replica of the transmitted waveform. For this derivation, a synchronous subcarrier, PCM signal source is assumed. A white, additive, gaussian noise channel and a loss-less symbol correlator are also assumed. Figure A1 is a diagram of the communications system model. A loss-less, linear, RF modem is assumed but is not shown explicitly. It is not really necessary to include the subcarrier modem since it does not affect the results. However, it is included just to emphasize that point.

The following definitions apply to this appendix as well as the body of the paper.

$x(t)$ = the serial NRZ-PCM binary digit source signal,

f_s = the subcarrier clock which is synchronous with $x(t)$,

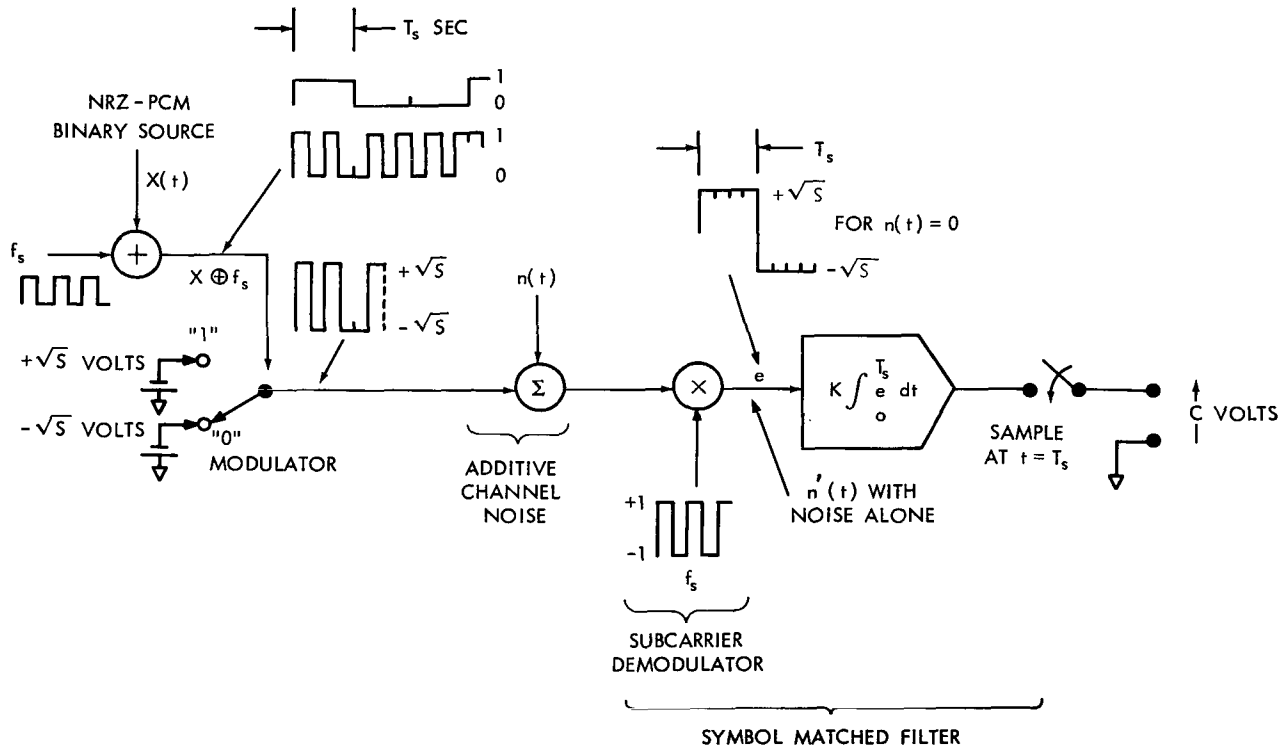


Figure A1—Signal, channel, and matched-filter models.

T_s = duration of a binary symbol x in seconds,

S = signal power in watts,

$n(t)$ = the additive channel noise; it is stationary, white, and gaussian with spectral density N_0 watts per Hz, and has zero mean,

K = the integrator's gain constant in sec^{-1} , and

C = the symbol correlation voltage at $t \geq T_s$ in volts.

Because the noise is additive and the filter is linear, the signal and noise effects on C may be found separately and superposition applied.

Signal Only

If $n(t) = 0$, then C is deterministic for each x . From the model in Figure A1, it is clear that for the 2-cycle-per-symbol subcarrier shown or for any " n "-cycle-per-symbol PCM subcarrier, " e " at the integrator input will always be of the same form ($n = 1, 2, 3, \dots$). Thus

$$C]_{x=1} = K \int_0^{T_s} \sqrt{S} dt = K \sqrt{S} T_s, \quad (\text{A1})$$

and

$$C]_{x=0} = K \int_0^{T_s} -\sqrt{S} dt = -K \sqrt{S} T_s. \quad (\text{A2})$$

Noise Only

The autocorrelation function of white, gaussian noise will be needed in the derivation below. It is well known that the autocorrelation function and the power spectral density are a Fourier transform pair. Therefore,

$$\phi(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} P(f) e^{+j\omega\tau} df, \quad (\text{A3})$$

where

$\phi(\tau)$ = the autocorrelation function of some random signal r , and

$P(f)$ = the power spectral density of the random signal;

$\phi(\tau)$ is defined here as

$$\phi(\tau) = E[n(t) n(t - \tau)] , \quad (A4)$$

where $n(t)$ is the random signal and $E(\)$ denotes mathematical expectation over an ensemble.

The autocorrelation function of $n(t)$ may be found from Equation A3 since $P(f) = N_0$ is known. Thus,

$$\begin{aligned} \phi_n(\tau) &= \frac{1}{2} \int_{-\infty}^{\infty} N_0 e^{+j\omega\tau} d\omega \\ &= \frac{N_0}{2} \delta(\tau) , \end{aligned} \quad (A5)$$

where $\delta(t)$ is the Dirac delta function.

Now it is desired to calculate the probability density function for the "noise alone" matched-filter output. The effects of the subcarrier multiplier will be considered first. The effect of the subcarrier multiplier on $n(t)$ may be found formally by using the convolution theorem. However, it can be simply observed that since the multiplier only multiplies by +1 and -1 and $n(t)$ is an infinite bandwidth signal, then the amplitude statistics of e , the product, are unchanged. The power spectral density N_0 is also unchanged. The noise time-sequence output from the multiplier $n'(t)$ is different from $n(t)$, but its amplitude statistics and power spectrum are unchanged.

The effects of the integrator may now be treated. Since the integrator is a linear summation device and the input noise $n'(t)$ has gaussian amplitude probability density, then the output C must also be gaussian. Knowing the mean and variance of C will then completely specify its statistics.

Define C_N = the matched-filter output with noise $n(t)$ alone as input. Then,

$$\begin{aligned} \text{mean}(C_N) &\triangleq E(C_N) \\ &= E \left[K \int_0^{T_s} n'(t) dt \right] , \end{aligned}$$

and since the expectation is computed over an ensemble,

$$\text{mean}(C_N) = K \int_0^{T_s} E[n'(t)] dt = 0 , \quad (A6)$$

where

$E(\) =$ mathematical expectation over an ensemble of events.

The variance of C_N is by definition:

$$\text{var} (C_N) = E[(C_N - \text{mean})^2]$$

and

$$\text{var} (C_N) = E(C_N^2)$$

here, and

$$\text{var} (C_N) = E \left[K \int_0^{T_s} n'(t) dt \cdot K \int_0^{T_s} n'(t) dt \right] .$$

The integrals may be combined provided that the integrand parameters are made distinct. Therefore

$$\begin{aligned} \text{var} (C_N) &= E \left[K^2 \int_0^{T_s} \int_0^{T_s} n(u) n(t) du dt \right] \\ &= K^2 \int_0^{T_s} \int_0^{T_s} E[n(u) n(t)] du dt ; \end{aligned}$$

but from Equations A4 and A5,

$$E[n(t) n(t - \tau)] = \frac{N_0}{2} \delta(\tau) .$$

Then,

$$\begin{aligned} \text{var} (C_n) &= K^2 \int_0^{T_s} \int_0^{T_s} \frac{N_0}{2} \delta(t - u) du dt \\ &= \frac{K^2 N_0}{2} \int_0^{T_s} dt = \frac{K^2 N_0 T_s}{2} . \end{aligned} \tag{A7}$$

Filter Output Statistics With Signal and Noise

With combined signal and noise entering the matched filter,

$$\text{mean}(C) = \text{mean}(\text{signal}) + \text{mean}(\text{noise})$$

and

$$K\sqrt{S} T_s \quad \text{for } x = 1 \quad (\text{A8})$$

$$- K\sqrt{S} T_s \quad \text{for } x = 0 . \quad (\text{A9})$$

The density of C is, of course, gaussian with variance given by Equation A7. A useful additional result is the mean-to-standard deviation ratio of C:

$$\frac{\mu_1}{\sigma} = \frac{K\sqrt{S} T_s}{\sqrt{\frac{K^2 N_0 T_s}{2}}} = \sqrt{2 \frac{S T_s}{N_0}} , \quad (\text{A10})$$

where

μ_1 = mean of C given $x = 1$, and

σ = standard deviation of C = $[\text{var}(C)]^{1/2}$.

Appendix B

Tabulation of the Symbol Mutual Information Function

This appendix is a tabulation of the symbol mutual information function $I(x = 1, C)$ for a number of symbol signal-to-noise ratios. This function, or approximations to it, may then be used to find the "log-a-posteriori probability (LAP) metric" for use in sequential decoding of binary PCM signals.

By definition,

$$I(x = 1, C) = \log_2 \frac{p(C | x = 1)}{p(C)} ; \quad (B1)$$

for the gaussian channel,

$$I(x = 1, C) = 1 - \log_2 \left(\epsilon^{-4C ST_s/N_0} + 1 \right) , \quad (B2)$$

where

x = the transmitted binary symbol, 0 or 1,

C = the symbol correlation at the output of a matched filter; $E(C | x = 1) = 1$ volt is assumed,

S = transmitted modulation power in watts,

T_s = duration of a symbol in sec, and

N_0 = noise density in watts per Hz.

Because of symmetry, it follows that

$$I(x = 0, C) = I(x = 1, -C) ;$$

therefore, only Equation B2 is tabulated in Table B1.

Finally, the LAP metric is found as

$$M_c(x | C) = I(x, C) - \frac{1}{V} ,$$

where

v = the number of binary symbols transmitted by the encoder for each information bit entered into the encoder.

Table B1
I(x = 1, C) vs. C.

C in Volts	ST _s /N ₀ in db; db = 10 log ₁₀ (ST _s /N ₀)									
	-6	-5	-4	-3	-2	-1	0	1	2	3
3.00	0.931	0.968	0.988	0.996	0.999	1.000	1.000	1.000	1.000	1.000
2.80	0.916	0.959	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000
2.60	0.898	0.947	0.977	0.992	0.998	1.000	1.000	1.000	1.000	1.000
2.40	0.876	0.932	0.969	0.988	0.997	0.999	1.000	1.000	1.000	1.000
2.20	0.850	0.913	0.957	0.983	0.994	0.999	1.000	1.000	1.000	1.000
2.00	0.818	0.889	0.941	0.974	0.991	0.997	1.000	1.000	1.000	1.000
1.80	0.781	0.859	0.920	0.961	0.985	0.995	0.999	1.000	1.000	1.000
1.60	0.736	0.821	0.891	0.943	0.975	0.991	0.998	1.000	1.000	1.000
1.40	0.683	0.773	0.852	0.915	0.958	0.983	0.995	0.999	1.000	1.000
1.20	0.621	0.714	0.801	0.875	0.932	0.968	0.988	0.997	0.999	1.000
1.00	0.549	0.641	0.732	0.817	0.889	0.941	0.974	0.991	0.997	1.000
0.80	0.465	0.552	0.644	0.735	0.820	0.891	0.942	0.975	0.991	0.998
0.60	0.369	0.445	0.530	0.621	0.713	0.800	0.875	0.931	0.968	0.988
0.40	0.260	0.318	0.387	0.465	0.551	0.643	0.734	0.819	0.890	0.942
0.20	0.136	0.170	0.210	0.259	0.318	0.386	0.464	0.550	0.642	0.733
-0.00	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
-0.20	-0.154	-0.196	-0.250	-0.320	-0.412	-0.532	-0.692	-0.905	-1.190	-1.573
-0.40	-0.321	-0.413	-0.534	-0.694	-0.908	-1.194	-1.578	-2.092	-2.774	-3.671
-0.60	-0.502	-0.652	-0.851	-1.118	-1.475	-1.955	-2.593	-3.435	-4.528	-5.932
-0.80	-0.696	-0.910	-1.197	-1.582	-2.098	-2.782	-3.682	-4.847	-6.338	-8.229
-1.00	-0.903	-1.187	-1.569	-2.079	-2.758	-3.650	-4.806	-6.286	-8.163	-10.533
-1.20	-1.121	-1.480	-1.961	-2.601	-3.445	-4.541	-5.948	-7.735	-9.994	-12.839
-1.40	-1.349	-1.786	-2.369	-3.140	-4.147	-5.445	-7.097	-9.188	-11.825	-15.146
-1.60	-1.587	-2.104	-2.791	-3.692	-4.860	-6.355	-8.250	-10.643	-13.657	-17.452
-1.80	-1.833	-2.431	-3.222	-4.253	-5.580	-7.269	-9.405	-12.098	-15.489	-19.759
-2.00	-2.086	-2.766	-3.661	-4.820	-6.303	-8.185	-10.560	-13.553	-17.321	-22.065
-2.20	-2.344	-3.108	-4.105	-5.391	-7.029	-9.102	-11.716	-15.009	-19.154	-24.372
-2.40	-2.609	-3.455	-4.554	-5.964	-7.756	-10.020	-12.872	-16.464	-20.986	-26.678
-2.60	-2.877	-3.805	-5.006	-6.540	-8.484	-10.938	-14.028	-17.919	-22.818	-28.985
-2.80	-3.149	-4.159	-5.460	-7.116	-9.213	-11.856	-15.184	-19.374	-24.650	-31.291
-3.00	-3.425	-4.516	-5.915	-7.694	-9.942	-12.774	-16.340	-20.830	-26.482	-33.598

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